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# STOCHASTIC PROGRAMMED SYNTHESIS IN A DIFFERENTIAL GAME WITH INTEGRAL PAYOFF *

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The problem of assured optimal control of a system subjected to infinite interference is formalized as a positional differential game /1/ in the typical case of integral index of the transient quality, and is solved by the method of stochastic program synthesis /2/, which is further developed in the present paper. An important feature here is the functional nature of the ancilliary stochastic construction considered that enables us to calculate the value of the game as the quality of a properly designed programmed stochastic maximum. Using the known value of the game, the optimal control action is determined using the method of extremal displacement to the so-called accompanying point. The results obtained here open the way for investigating functional game problems of control in irregular cases.

## 1. Statement of the problem.

Consider the system

$$x^{\cdot} = A(t)x + B(t)u + C(t)v, \quad t_0 \leqslant t \leqslant \vartheta, \quad u \in Q, \quad v \in R \tag{1.1}$$

where $t_0$ and $\vartheta$ are fixed instants of time, $x \in E^n$ is the phase vector of the object, $u$ is the control, $v$ is interference, $Q \subset E^q$, $R \subset E^r$ are convex compacta, and $A(t)$, $B(t)$, $C(t)$ are continuous matrix functions.

Let the functional

$$\gamma = \gamma(x[t_*[\cdot]\vartheta]) = \Big(\int\limits_{[t_*,\vartheta]} |D(t)[x[t] - y(t)]|^2 \mu(dt)\Big)^{1/2} \tag{1.2}$$

$$x[t_*[\cdot]\vartheta] = \{x[t], \quad t_* \leqslant t \leqslant \vartheta\}, \quad t_* \in [t_0, \vartheta]$$

be given, where $D(t)$ is a piecewise continuous matrix function, $y(t)$ is a piecewise continuous vector function, $|z|$ is the Euclidean norm of the vector $z$, $\mu(dt)$ is a measure which for any segment $[\tau_*, \tau^*] \subset [t_0, \vartheta]$ has the form

$$\mu([\tau_*, \tau^*]) = \sum_p \mu(\nu_p) + \int\limits_{\tau_*}^{\tau^*} \eta(t)\,dt, \quad \mu(\nu_p) > 0 \tag{1.3}$$

where $\eta(t) \geqslant 0$, $t_0 \leqslant t \leqslant \vartheta$ is a piecewise-continuous function, and $\nu_p$ are points of the segment $[\tau_*, \tau^*]$ that belong to a given finite set of points specified on segments $_0, \vartheta]$.

The problem is to determine the law of control which forms the action $u = u[t]$ based on information on the current position of the object (1.1), and ensures the lowest possible value of the $\gamma$ index (1.2). It is then possible to come across any measurable interference $v[\cdot] = \{v[t] \in R, \ t_* \leqslant t < \vartheta\}$. Thus the problem can be conveniently included in a differential game in which the place of the second player is assigned to Nature. The input problem is then supplemented by the problem of deriving the control law that forms the action $v = v[t]$ also on the feedback principle, and ensures the maximum possible value of $\gamma$ (1.2) on the motion of the object.

A rigorous mathematical formulation of this game is given in /1-3/. We merely note here that in conformity with the definition /1/ the functional $\gamma$ (1.2) is positional. In this case the value of the differential game is $\rho^\circ = \rho^\circ(t, x)$ and it has a saddle point $\{u^\circ(\cdot), v^\circ(\cdot)\}$ which is formed by a pair of universal strategies

$$u^\circ(\cdot) = \{u^\circ(t, x, \varepsilon) \in Q, \ x \in E^n, \ t_0 \leqslant t \leqslant \vartheta, \ \varepsilon > 0\}$$
$$v^\circ(\cdot) = \{v^\circ(t, x, \varepsilon) \in R, \ x \in E^n, \ t_0 \leqslant t \leqslant \vartheta, \ \varepsilon > 0\}$$

where $\varepsilon$ is the parameter of exactitude. The strategies $u^\circ(\cdot)$ and $v^\circ(\cdot)$ are effectively constructed by the method of extremal displacement of the object (1.1) to the so-called accompanying point, using the function $\rho^\circ = \rho^\circ(t, x)$.

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A formula for calculating the value of the game in the case of functional $\gamma$ (1.2) is given below, and expresses the coincidence of the latter in any arbitrary position $\{t_*, x_*\}$ with the value of the properly designed stochastic programmed maximum.

2. **Explanatory example.** Suppose the object moves along the horizontal axis $\zeta$ under the action of a thrust u and interference $v$ (e.g., the force of the wind), in accordance with the differential equations

$$x_1^{\cdot} = x_2, \; x_2^{\cdot} = u + v, \; |u| \leqslant \lambda_1, \; |v| \leqslant \lambda_2, \; t_0 \leqslant t \leqslant \vartheta \tag{2.1}$$

where $x_1 = \zeta$ is the coordinate and $x_2 = \zeta^{\cdot}$ is the velocity of the object. Let the instants $\nu_1 \in (t_0, \vartheta), \nu_2 \in (\nu_1, \vartheta)$ and the points $\zeta_1, \zeta_2, \zeta_3$ be indicated on the $\zeta$ axis. It is required to determine the control effort so that the object (2.1) is, at instants $\nu_1$ and $\nu_2$, as close as possible to the points $\zeta_1$ and $\zeta_2$, respectively, and at the instant $\vartheta$ as close as possible to the points $\zeta_3$, having then a low velocity $x_2[\vartheta]$. Moreover the object is to move in the time interval $(\nu_1, \nu_2)$ at the velocity $x_2[t]$ that is close to the specified $c = (\zeta_2 - \zeta_1)/(\nu_2 - \nu_1)$. Let us assume that the deviation from these requirements incurs the penalty

$$\gamma = \Big[\mu_1 (x_1[\nu_1] - \zeta_1)^2 + \mu_2 (x_1[\nu_2] - \zeta_2)^2 + \mu_3 [(x_1[\vartheta] - \zeta_3)^2 + x_2^2[\vartheta]] + \mu_4 \int_{\nu_1}^{\nu_2} (x_2[t] - c)^2 \, dt\Big]^{1/2} \tag{2.2}$$

where $\mu_i > 0, \; i = 1, \ldots, 4$ are weighting factors.

Let the initial position be $t_* \in [t_0, \vartheta], \; t_* < \nu_1, \; x_{1*} = \zeta_*, \; x_{2*} = \zeta_*^{\cdot}$. It is natural to formulate the problem of the optimal strategy $u^\circ(\cdot) = u^\circ(t, x_1, x_2, \varepsilon)$ which will ensure the lowest possible value of the index $\gamma$ (2.2). However, that index takes the form of the functional $\gamma$ (1.2), (1.3), if we assume

$$x[t] = \{x_1[t], \; x_2[t]\}, \; \mu(\nu_1) = \mu_1, \; \mu(\nu_2) = \mu_2, \; \mu(\vartheta) = \mu_3; \; \eta(t) \equiv \mu_4, \; t \in [\nu_1, \; \nu_2],$$
$$\eta(t) \equiv 0, \; t \in [t_0, \; \vartheta] \setminus [\nu_1, \nu_2]; \; y(t) = \{y_1(t), y_2(t)\}, \; y_1(t) \equiv \zeta_1, \; t_0 \leqslant t \leqslant \nu_1, \; y_1(t) \equiv$$
$$\zeta_2, \; \nu_1 < t \leqslant \nu_2, \; y_1(t) \equiv \zeta_3, \; \nu_2 < t \leqslant \vartheta; \; y_2(t) \equiv c, \; t \in (\nu_1, \; \nu_2), \; y_2(t) \equiv 0, \; t \in [t_0, \vartheta] \setminus (\nu_1, \nu_2)$$
$$D(t) = \begin{Vmatrix} 1 & 0 \\ 0 & 0 \end{Vmatrix}, \; t_0 \leqslant t \leqslant \nu_1; \; D(t) = \begin{Vmatrix} 0 & 0 \\ 0 & 1 \end{Vmatrix}, \; \nu_1 < t < \nu_2$$
$$D(\nu_2) = \begin{Vmatrix} 1 & 0 \\ 0 & 0 \end{Vmatrix}; \; D(t) = \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix}, \; \nu_2 < t \leqslant \vartheta$$

and supplement the definition of $\gamma$ (2.2) for values $t_* \geqslant \nu_2$, following the method of introducing the differential game for a positional functional.

The required optimal strategy $u^\circ = u^\circ(t, x_1, x_2, \varepsilon)$ exists, according to /1/. If the motion of the object (2.1) is formulated on the basis of this in a discrete scheme with a step not exceeding $\delta > 0$, it ensures for any interference $v[\cdot] = \{|v[t]| \leqslant \lambda_2, \; t_* \leqslant t < \vartheta\}$ a value of $\gamma$ (2.2) not greater than $\rho^\circ(t_*, x_{1*}, x_{2*}) + \chi$, where $\chi > 0$ is arbitrarily small, when $\varepsilon$ and $\delta$ are fairly small.

3. **The stochastic programmed maximin.** Let us fix some arbitrary position $\{t_*, x_*\}, \; t_* \in [t_0, \vartheta], \; x_* \in E^n$, select the partitioning $\Delta_k\{t_j\}, \; t_1 = t_*, \; t_j < t_{j+1}, \; t_{k+1} = \vartheta$ of the segment $[t_*, \vartheta]$, and attach to them a set of random quantities $\xi_j, \; j = 1, \ldots, k$ each of which is uniformly distributed over the half-interval $0 \leqslant \xi_j < 1$. Each set $\{\xi_1, \ldots, \xi_k\}$ may be treated as an elementary event $\omega$ from the probability space $\{\Omega, F, P\}$, where $\Omega = \{\omega\}$ is a unit $k$-dimensional cube in space $\{\xi_1, \ldots, \xi_k\}$, $F$ is a Borel $\sigma$-algebra for that cube, and $P(B)$, and $B \in F$ is the Lebesgue measure on that cube /4/.

We call the vector functions $u(\cdot) = \{u(t, \; \omega) \in Q, \; t_* \leqslant t < \vartheta, \; \omega \in \Omega\}$ and $v(\cdot) = \{v(t, \; \omega) \in R, \; t_* \leqslant t < \vartheta, \; \omega \in \Omega\}$ specified in the half-intervals $[t_j, \; t_{j+1}), \; j = 1, \ldots, k$ by the equations

$$u(t, \; \omega) = u[t, \; \xi_1, \ldots, \xi_j], \; v(t; \; \omega) = v[t, \; \xi_1, \ldots, \xi_j] \tag{3.1}$$

in which the functions $u = u[t, \xi_1, \ldots, \xi_j]$ and $v = v[t, \xi_1, \ldots, \xi_j]$ are Borel measurable over the totality of arguments $t, \xi_1, \ldots, \xi_j$, the stochastically non-predicting programmes $u(\cdot)$ and $v(\cdot)$.

Consider the stochastic differential equation

$$w^{\cdot} = A(t) w + B(t) u(t, \; \omega) + C(t) v(t, \; \omega), \; w[t_*] = x_* \tag{3.2}$$

where $\{u(\cdot), \; v(\cdot)\}$ is some pair of programs (3.1). The solution of this equation is given by the Cauchy formula

$$w(t, \omega, u(\cdot), v(\cdot)) = X(t, t_*) x_* + \int_{t_*}^{t} X(t, \tau)[B(\tau) u(\tau, \omega) + C(\tau) v(\tau, \omega)] \, d\tau, \; t_* \leqslant t \leqslant \vartheta \quad \omega \in \Omega \tag{3.3}$$

where $X(t, \; t_*)$ is the fundamental matrix of the respective homogeneous differential equation. In consequence of (3.1), when $t_j \leqslant t < t_{j+1}$, the following equations hold:

$$w\,(t,\ \omega,\ u\,(\cdot),\ v\,(\cdot)) = w\,[t,\ \xi_1,\ \ldots,\ \xi_j,\ u\,(\cdot),\ v\,(\cdot)] \tag{3.4}$$

where the functions on the right side are Borel measurable over the set of arguments $\xi_1, \ldots, \xi_j$.

The functions $w\,(\cdot,\ u\,(\cdot),\ v\,(\cdot))$ (3.3) may be considered as elements of a Hilbert space $L^{(2)}\,\{[t_*,\ \vartheta] \times \Omega\}$ of random functions /5/ with the scalar product

$$(w^{(1)}\,(\cdot), w^{(2)}\,(\cdot)) = \int\limits_{\Omega}\int\limits_{[t_*,\,\vartheta]} \langle w^{(1)}\,(t, \omega)\cdot w^{(2)}\,(t, \omega)\rangle\, \mu\,(dt)\, P\,(d\omega) \tag{3.5}$$

and the norm

$$\|\, w\,(\cdot)\,\| = \Big(\int\limits_{\Omega}\int\limits_{[t_*,\,\vartheta]} |\, w\,(t, \omega)|^2\, \mu\,(dt)\, P\,(d\omega)\Big)^{1/2}$$

where $\langle w^{(1)}\cdot w^{(2)}\rangle$ is the scalar product in $E^n$.

For a given initial position $\{t_*,\ x_*\}$ and fixed partitioning $\Delta_k = \Delta_k\,\{t_j\}$ we determine the stochastic programmed maximin as the quantity

$$\rho\,(t_*, x_*, \Delta_k) = \sup_{v(\cdot)}\, \min_{u(\cdot)} \Big(\int\limits_{\Omega}\int\limits_{[t_*,\,\vartheta]} |\, D\,(t)[w\,(t, \tag{3.6}$$

$$\omega,\ u\,(\cdot),\ v\,(\cdot)) - y\,(t)]\ |^2 \mu\,(dt) P\,(d\omega)\Big)^{1/2}$$

We assume that the partitioning $\Delta_k$ satisfies the condition $\max_j\ (t_{j+1} - t_j) \leqslant \delta\,(k) = [\vartheta - t_0]/k$.

*Theorem.* Whatever the sequence of partitioning $\{\Delta_k,\ k = 1,\ 2,\ \ldots\}$ the limit

$$\lim_{k\to\infty} \rho\,(t_*,\ x_*,\ \Delta_k) = \rho^*\,(t_*,\ x_*) \tag{3.7}$$

exists, which is independent of the selection of the partitioning, and agrees with the value of the differential game $\rho^\circ\,(t_*, x_*)$ whatever the position of $\{t_*,\ x_*\}$.

## 4. The subsidiary quantities.

The proof of the theorem consists of several steps. We select the partitioning $\Delta_k\,\{t_j\}$ and assume that $t^* = t_r$ is one of the points of this partitioning. We fix in the half-interval $[t_*,\ t^*)$ a pair of measurable functions

$$u\,[\cdot] = u\,[t_*[\cdot]\,t^*) = \{u\,[t] \in Q,\ t_* \leqslant t < t^*\}$$
$$v\,[\cdot] = v\,[t_*[\cdot]\,t^*) = \{v\,[t] \in R,\ t_* \leqslant t < t^*\}$$

which we shall call actions. Consider the set $U\,[t^*] = \{u\,(\cdot)\}$ of stochastic non-predicting programs $u\,(\cdot)$, each of which is composed of fixed $u\,[\cdot]$ and any arbitrary stochastic non-predicting program (3.1) of the form

$$u^*\,(t,\ \omega) = u\,[t,\ \xi_r,\ \ldots,\ \xi_j],\quad t_j \leqslant t < t_{j+1},\ \ j = r,\ \ldots,\ k$$

such that $u\,(\cdot) = \{u\,[\cdot],\ u^*\,(\cdot)\}$. We similarly construct the set $V\,[t^*] = \{v\,(\cdot) = \{v\,[\cdot],\ v^*\,(\cdot)\}\}$.

Let $l\,(\cdot) = \{l\,(t,\ \omega),\ t_* \leqslant t \leqslant \vartheta,\ \omega \in \Omega\}$ be an arbitrary element of space $L^{(2)}\,\{[t_*, \vartheta] \times \Omega\}$ corresponding to the partitioning $\Delta_k$. We denote by $L\,[t^*]$ the set of elements $l\,(\cdot)$ of the form $l\,(t, \omega) = \{l\,[t,\ \xi_r,\ \ldots,\ \xi_k],\ t_* \leqslant t \leqslant \vartheta,\ \omega \in \Omega\}$ constrained by the condition

$$\Big(\int\limits_{\Omega}\int\limits_{[t_*,\,\vartheta]} |\, D\,(t)\, l\,(t, \omega)\, |^2\, \mu\,(dt)\, P\,(d\omega)\Big)^{1/2} \leqslant 1 \tag{4.1}$$

Let $w\,(\cdot,\ u\,(\cdot),\ v\,(\cdot))$ be the motion (3.3) generated by some pair of programs $u\,(\cdot) \in U\,[t^*]$, $v\,(\cdot) \in V\,[t^*]$ and let

$$w^{(1)}\,(\cdot,\ u\,(\cdot),\ v\,(\cdot)) = \{w^{(1)}\,(t,\ \omega,\ u\,(\cdot),\ v\,(\cdot)) = D\,(t)[w\,(t,\ \omega,$$
$$u\,(\cdot),\ v\,(\cdot)) - y\,(t)],\ t_* \leqslant t \leqslant \vartheta,\ \omega \in \Omega\}$$

Let us take some element $l\,(\cdot) \in L\,[t^*]$ and consider the scalar product $(w^{(1)}\,(\cdot,\ u\,(\cdot),\ v\,(\cdot)),\ l^{(1)}\,(\cdot))$, where $l^{(1)}\,(\cdot) = \{l^{(1)}\,(t,\ \omega) = D\,(t) l\,(t,\ \omega),\ t_* \leqslant t \leqslant \vartheta,\ \omega \in \Omega\}$. By (3.3) and (3.5) we have

$$(w^{(1)}\,(\cdot, u\,(\cdot), v\,(\cdot)), l^{(1)}\,(\cdot)) = \int\limits_{\Omega}\int\limits_{[t_*,\,\vartheta]} \langle D\,(t)\, l\,(t, \omega)\cdot D\,(t)\, X\,(t, t_*)\, x_*\rangle\, \mu\,(dt)\, P\,(d\omega) + \tag{4.2}$$

$$\int\limits_{\Omega}\int\limits_{[t_*,\,\vartheta]} \Big\langle D\,(t)\, l\,(t, \omega)\cdot \int\limits_{t_*}^{t} D\,(t)\, X\,(t, \tau)[B\,(\tau)\, u\,(\tau, \omega) + C\,(\tau)\, v\,(\tau, \omega)]\, d\tau\Big\rangle\, \mu\,(dt)\, P\,(d\omega) -$$

$$\int\limits_{\Omega}\int\limits_{[t_*,\,\vartheta]} \langle D\,(t)\, l\,(t, \omega)\cdot D\,(t)\, y\,(t)\rangle\, \mu\,(dt)\, P\,(d\omega)$$

We change the order of integration with respect to $\mu(dt)$ and $d\tau$ in the second term on the right side of (4.2), and put

$$G(t) = D'(t) D(t), \quad M\{l(t, \omega)\} = m(t) \tag{4.3}$$
$$\psi[\tau] = \int_{[\tau, \vartheta]} X'(t, \tau) G(t) m(t) \mu(dt)$$

$$\psi[\tau, \xi_r, \ldots, \xi_j] = M\Big\{\int_{[\tau, \vartheta]} X'(t, \tau) G(t) l[t, \xi_r, \ldots, \xi_k] \mu(dt) \,|\, \xi_r, \ldots, \xi_j\Big\} \tag{4.4}$$

where $M\{\ldots\}$ is the expectation and $M\{\ldots|\ldots\}$ is the conditional expectation. The prime denotes transposition. Taking into account the program structure $u(\cdot) \in U[t^*]$, $v(\cdot) \in V[t^*]$ and the notation (4.3) and (4.4), we obtain

$$\begin{aligned}(w^{(1)}(\cdot, u(\cdot), v(\cdot)), l^{(1)}(\cdot)) = \langle \psi[t_*] \cdot x_* \rangle + \\ \int_{t_*}^{t^*} \langle \psi[\tau] \cdot (B(\tau) u[\tau] + C(\tau) v[\tau]) \rangle d\tau + \\ \sum_{j=r}^{k} \int_{t_j}^{t_{j+1}} M\{\langle \psi[\tau, \xi_r, \ldots, \xi_j] \cdot (B(\tau) u[\tau, \xi_r, \ldots, \xi_j] + \\ C(\tau) v[\tau, \xi_r, \ldots, \xi_j]) \rangle\} d\tau - \int_{[t_*, \vartheta]} \langle G(t) m(t) \cdot y(t) \rangle \mu(dt)\end{aligned} \tag{4.5}$$

Functions $\psi[\tau]$, $t_* \leqslant \tau \leqslant t^*$; $\psi[\tau, \xi_r, \ldots, \xi_j]$, $t_j \leqslant \tau < t_{j+1}$, $j = r, \ldots, k$ are piecewise continuous in $\tau$, and according to (1.3) may have first-order discontinuities only at the points $\tau = \nu_p$. The functions $\psi[\tau, \xi_r, \ldots, \xi_j]$ are measurable over the set of arguments $\tau, \xi_r, \ldots, \xi_j$. For the functions $\psi[\tau]$ in any interval $(\tau_*, \tau^*) \subset [t_*, t^*]$ that does not contain its points of discontinuity we have the estimate

$$|\psi[\tau^*] - \psi[\tau_*]| \leqslant g[\tau^* - \tau_*]^{1/2}, \quad g = \text{const} \tag{4.6}$$

We introduce, based on (4.5), the quantity

$$\begin{aligned}\varkappa[t^*] = \varkappa(t_*, x_*, t^*, u[t_*[\cdot] t^*), v[t_*[\cdot] t^*), \\ \Delta_k, l(\cdot)) = \langle \psi[t_*] \cdot x_* \rangle + \\ \int_{t_*}^{t^*} \langle \psi[\tau] \cdot (B(\tau) u[\tau] + C(\tau) v[\tau]) \rangle d\tau + \\ \sum_{j=r}^{k} \sum_{t_j}^{t_{j+1}} M\{\max_{v \in R} \min_{u \in Q} \langle \psi[\tau, \xi_r, \ldots, \xi_j] \cdot (B(\tau) u + \\ C(\tau) v) \rangle\} d\tau - \int_{[t_*, \vartheta]} \langle G(t) m(t) \cdot y(t) \rangle \mu(dt)\end{aligned} \tag{4.7}$$

Owing to the properties of the functions (4.3) and (4.4), the quantity $\varkappa[t^*]$ has been correctly defined. The same properties enable us to establish for the quantity $\varkappa[t^*]$, when $t^* = t_*$, the following equations:

$$\begin{aligned}\varkappa[t_*] = \langle \psi[t_*] \cdot x_* \rangle + \\ \sum_{j=1}^{k} \int_{t_j}^{t_{j+1}} M\{\max_{v \in R} \min_{u \in Q} \langle \psi[\tau, \xi_1, \ldots \xi_j] \cdot (B(\tau) u + \\ C(\tau) v) \rangle\} d\tau - \int_{[t_*, \vartheta]} \langle G(t) m(t) \cdot y(t) \rangle \mu(dt) = \\ \max_{v(\cdot)} \min_{u(\cdot)} (w^{(1)}(\cdot, u(\cdot), v(\cdot)), l^{(1)}(\cdot))\end{aligned} \tag{4.8}$$

Using $\varkappa[t^*]$ of (4.7), we construct the quantity

$$\begin{aligned}\varphi[t^*] = \varphi(t_*, x_*, t^*, u[t_*[\cdot] t^*), v[t_*[\cdot] t^*), \Delta_k) = \\ \sup_{l(\cdot) \in L[t^*]} \varkappa(t_*, x_*, u[t_*[\cdot] t^*), v[t_*[\cdot] t^*), \Delta_k, l(\cdot))\end{aligned} \tag{4.9}$$

If we include here the case of $t^* = \vartheta$, all components that define $\varphi[t^*]$ when $t^* = \vartheta$ are functions only of $t$. Let $t_i$ and $t_{i+1}$, $i \in \{1, \ldots, k-1\}$, be two consequtive arbitrary instants of partitioning $\Delta_k = \Delta_k\{t_i\}$.

Let us evaluate the remainder

$$\Delta\varphi_i = \varphi[t_{i+1}] - \varphi[t_i] \tag{4.10}$$

assuming that the actions $u[t_*[\cdot]t_{i+1})$ and $v[t_*[\cdot]t_{i+1})$ which determine $\varphi[t_{i+1}]$ are formed from the actions $u[t_*[\cdot]t_i)$ and $v[t_*[\cdot]t_i)$ that determine the quantity $\varphi[t_i]$, and certain actions $u[\cdot]=u[t_i[\cdot]t_{i+1})$ and $v[\cdot]=v[t_i[\cdot]t_{i+1})$. Let $\{l^{(s)}(\cdot)\in L[t_{i+1}],\ s=1,2,\ldots\}$ be some maximizing sequence for the quantity $\varphi[t_{i+1}]$. It can then be shown that for the remainder $\Delta\varphi_i$ (4.10) the following estimate holds:

$$\Delta\varphi_i\leqslant\int_{t_i}^{t_{i+1}}[\langle\psi^{(s)}[\tau]\cdot B(\tau)u[\tau]\rangle- \tag{4.11}$$

$$\min_{u\in Q}\langle\psi^{(s)}[\tau]\cdot B(\tau)u\rangle]\,d\tau+\varepsilon_s\,(\varepsilon_s\to 0\ \text{при}\ s\to\infty)$$

$$\psi^{(s)}[\tau]=\int_{[\tau,\vartheta]}X'(t,\tau)G(t)M\{l^{(s)}[t,\xi_{i+1},\ldots,\xi_k]\}\mu(dt) \tag{4.12}$$

Because of (4.1) the functions $\psi^{(s)}[\cdot]=\psi^{(s)}[t_i[\cdot]t_{i+1}]$, $s=1,2,\ldots$ are uniformly bounded and owing to (4.6) are equicontinuous in any interval $(\tau_*,\tau^*)\subset[t_i,t_{i+1}]$ that does not contain points of discontinuity of these functions $\psi^{(s)}[\cdot]$ (4.12). However, all points of discontinuity of the functions $\psi^{(s)}[\cdot]$, $s=1,2,\ldots$ are contained in one finite set of points. Hence, it is possible to separate from the sequence $\{\psi^{(s)}[\cdot],\ s=1,2,\ldots\}$ a subsequence that converges uniformly in $\tau$ to some function $\psi^*[\cdot]$ at all points of the segment $[t_i,t_{i+1}]$. This enables us to obtain from (4.11) the estimate

$$\Delta\varphi_i\leqslant\int_{t_i}^{t_{i+1}}[\langle\psi^*[\tau]\cdot B(\tau)u[\tau]\rangle-\min_{u\in Q}\langle\psi^*[\tau]\cdot B(\tau)u\rangle]\,d\tau \tag{4.13}$$

which holds for any $i\in\{1,\ldots,k-1\}$ and any limit functions $\psi^*[\cdot]$. For the remainder $\Delta\varphi_k=\varphi[\vartheta]-\varphi[t_k]$ we have the same estimate (4.13) as for $\Delta\varphi_i$ (4.10), $i=1,\ldots,k-1$.

**5. The properties of the subsidiary quantities.** The following lemma establishes the basic property called the *u*-stability.

*Lemma.* Whatever the two consecutive instants $t_i<t_{i+1}$, $i\in\{1,\ldots,k\}$ of partitioning $\Delta_k\{t_j\}$ and whatever the actions $u[t_*[\cdot]t_i)$ and $v[t_*[\cdot]t_i)$, for any action $v[\cdot]=v[t_i[\cdot]t_{i+1})$ an action $u[\cdot]=u[t_i[\cdot]t_{i+1})$ can be found such that the inequality

$$\Delta\varphi_i=\varphi[t_{i+1}]-\varphi[t_i]\leqslant 0,\quad i=1,\ \ldots,\ k \tag{5.1}$$

is satisfied.

The following is the plan for proving the lemma. As a result of (4.13) it is sufficient to establish the existence of the action $u^\circ[\cdot]=\{u^\circ[\tau]\in Q,\ t_i\leqslant\tau<t_{i+1}\}$ which ensures the equation

$$\int_{t_i}^{t_{i+1}}[\langle\psi^*[\tau]\cdot B(\tau)u^\circ[\tau]\rangle-\min_{u\in Q}\langle\psi^*[\tau]\cdot B(\tau)u\rangle]\,d\tau=0 \tag{5.2}$$

To construct this action $u^\circ[\cdot]$ we use the theorem on the fixed point /6/. The set of all possible actions $u[\cdot]$ is bounded, convex, and weakly compact in $L^{(2)}\{[t_i,t_{i+1})\}$. When the action $v[\cdot]$ is fixed, to each action $u[\cdot]\in U$ there corresponds a set $\Psi^*(u[\cdot])$ of limit functions $\psi^*[\cdot]=\{\psi^*[\tau],\ t_i\leqslant\tau<t_{i+1}\}$ that are uniformly bounded and, because of (4.6), equi-continuous in any interval $(\tau_*,\tau^*)\subset[t_i,t_{i+1}]$ that does not contain points of discontinuity of the functions $\psi^*[\cdot]$. Hence the set $\Psi^*(u[\cdot])$ belongs to some convex compactum $\Psi$ in the space $L^{(2)}\{[t_i,t_{i+1})\}$. Moreover, it can be shown that the set $\Psi^*(u[\cdot])$ is convex and changes strongly semicontinuously from above on the inclusion relative to the change of $u[\cdot]$, which is weakly estimated. (Note that the property of convexity is ensured here by the stochastic properties of the structure).

We place in correspondence to each function $\psi[\cdot]\in\Psi$ the set $U^*(\psi[\cdot])$ of measurable functions $u^*[\cdot]=\{u^*[\tau]\in Q,\ t_i\leqslant\tau<t_{i+1}\}$ selected (according to the theorem on measurable choice) from the condition

$$\langle\psi[\tau]\cdot B(\tau)u^*[\tau]\rangle=\min_{u\in Q}\langle\psi[\tau]\cdot B(\tau)u\rangle \tag{5.3}$$

The set $U^*(\psi[\cdot])$ contained in $U$ is convex and changes weakly semicontinuously from above on the inclusion relative to a change of $\psi[\cdot]$ that is estimated strongly. Let us now consider the mapping of the set $S=\{s[\cdot]\}$ of all possible pairs $s[\cdot]=\{u[\cdot],\psi[\cdot]\}$, $u[\cdot]\in U$, $\psi[\cdot]\in\Psi$ into themselves, setting in correspondence to each pair $s[\cdot]\in S$ a non-empty set $\Phi(s[\cdot])\subset S$ of similar pairs $s^*[\cdot]=\{u^*[\cdot],\psi^*[\cdot]\}$, $u^*[\cdot]\in U^*(\psi[\cdot])$, $\psi^*[\cdot]\in\Psi^*(u[\cdot])$. The set $S$ is convex and weakly closed. The set $\Phi(s[\cdot])$ is convex weakly varying sequentially from above on the inclusion relative to the variation of $s[\cdot]$. Then in accordance with the theorem on the fixed point, an element $s^\circ[\cdot]\in S$ exists such that $s^\circ[\cdot]\in\Phi(s^\circ[\cdot])$, i.e. a $u^\circ[\cdot]$ and $\psi^\circ[\cdot]$ exist such that $u^\circ[\cdot]\in U^*(\psi^\circ[\cdot])$ and $\psi^\circ[\cdot]\in\Psi^*(u^\circ[\cdot])$. Hence the solution $u^\circ[\cdot]$ required ensuring the equation (5.2), is determined by the condition (5.3) in which it is necessary to set $\psi[\cdot]=\psi^\circ[\cdot]$. This proves the lemma.

When $t^* = t_*$, taking into account (4.8), we have the equation

$$\varphi[t_*] = \sup_{l(\cdot)\in L[t_*]} \max_{v(\cdot)} \min_{u(\cdot)} (w^{(1)}(\cdot,\ u(\cdot),\ v(\cdot)),\ l^{(1)}(\cdot)) \tag{5.4}$$

Since $(w^{(1)}(\cdot,\ u(\cdot),\ v(\cdot)),\ l^{(1)}(\cdot)) \leqslant \| w^{(1)}(\cdot,\ u(\cdot),\ v(\cdot)) \| \cdot \| l^{(1)}(\cdot) \|$, then in accordance with the definition of the quantity $\rho(t_*, x_*, \Delta_*)$ (3.6) and condition (4.1) we have

$$\varphi[t_*] \leqslant \sup_{l(\cdot)\in L[t_*]} \max_{v(\cdot)} \min_{u(\cdot)} [\| w^{(1)}(\cdot,\ u(\cdot),\ v(\cdot)) \| \cdot \| l^{(1)}(\cdot) \|] \leqslant \\ \max_{v(\cdot)} \min_{u(\cdot)} \| w^{(1)}(\cdot,\ u(\cdot),\ v(\cdot)) \| = \rho(t_*,\ x_*,\ \Delta_k) \tag{5.5}$$

When $t^* = \vartheta$, in accordance with (4.5) and (4.9) the quantity $\varphi[\vartheta]$ has the property

$$\varphi[\vartheta] = \sup_{l(\cdot)\in L[\vartheta]} (w^{(1)}(\cdot, u[\cdot], v[\cdot]), l^{(1)}(\cdot)) = \\ \| w^{(1)}(\cdot, u[\cdot], v[\cdot]) \| = \Big( \int_{[t_*, \vartheta]} | D(t)[w[t] - y[t]] |^2 \mu(dt) \Big)^{1/2} \tag{5.6}$$

**6. The stochastic maximin and the payoff of the game.** We select some partitioning sequence $\Delta_q\{t_j\}$, $q = 1, 2, \ldots$, for which the following limit exists:

$$\lim_{q\to\infty} \rho(t_*,\ x_*,\ \Delta_q) = \rho^*(t_*,\ x_*) \tag{6.1}$$

Then for any number $\alpha > 0$ a positive integer $k(\alpha)$ can be found such that for any $q \geqslant k(\alpha)$ the inequality

$$\rho(t_*,\ x_*,\ \Delta_q) \leqslant \rho^*(t_*,\ x_*) + \alpha \tag{6.2}$$

holds.

Consider $\{\varepsilon, \Delta_q\}$, the motion /1/ $x^\circ[t_*[\cdot]\vartheta]$ of the object (1.1) generated from the position $\{t_*, x_*\}$ by the partitioning steps $\Delta_q\{t_j\}$, $q \geqslant k(\alpha)$ of the optimal strategy $v^\circ(\cdot)$ of the second player and the application of a control $u[t_*[\cdot]\vartheta)$ composed of those sections $u[t_j[\cdot]t_{j+1})$, $j = 1, \ldots, q$, which, in conformity with the lemma, for actions $v^\circ[t_j[\cdot]t_{j+1})$ generated by the strategy $v^\circ(\cdot)$, ensure the satisfaction of the following inequalities:

$$\varphi(t_*, x_*, t_{j+1}, u[t_*[\cdot]t_{j+1}), v^\circ[t_*[\cdot]t_{j+1}), \Delta_q) \leqslant \varphi(t_*, x_*, t_j, u[t_*[\cdot]t_j) v^\circ[t_*[\cdot]t_j), \Delta_q)$$

From this chain of inequalities with $j = 1, \ldots, q$ and the properties (5.5) and (5.6) we obtain

$$\rho(t_*, x_*, \Delta_q) \geqslant \varphi[t_*] \geqslant \varphi[\vartheta] = \Big( \int_{[t_*, \vartheta]} | D(t)[x^\circ[t] - y(t)] |^2 \mu(dt) \Big)^{1/2} \tag{6.3}$$

But by the property of optimal strategy $v^\circ(t, x, \varepsilon)$ and the definition of the value of the game $\rho^\circ(t_*, x_*)$ /1/ we can indicate for any $\chi > 0$ such quantities $\delta(\chi) > 0$ and $\delta(\varepsilon, \chi) > 0$ that for any measurable application of the control $u[t_*[\cdot]\vartheta) = \{u[t] \in Q, t_* \leqslant t < \vartheta\}$ the inequality

$$\gamma(x^\circ[t_*[\cdot]\vartheta]) \geqslant \rho^\circ(t_*, x_*) - \chi \tag{6.4}$$

is satisfied so long as $\varepsilon \leqslant \varepsilon(\chi)$ and the partition step $\delta(q)$ of the segment $[t_*, \vartheta]$ is less than $\delta(\varepsilon, \chi)$. Hence selecting $\varepsilon \leqslant \varepsilon(\chi)$ and setting $q > \max\{k(\alpha), [\vartheta - t_0]/\delta(\varepsilon, \chi)\}$ in the partitioning $\Delta_q\{t_j\}$, we simultaneously have the inequalitites (6.2) and (6.3) and, by (6.4), the inequality

$$\Big( \int_{[t_*, \vartheta]} | D(t)[x^\circ[t] - y(t)] |^2 \mu(dt) \Big)^{1/2} \geqslant \rho^\circ(t_*, x_*) - \chi \tag{6.5}$$

Equating (6.2), (6.3) and (6.5), we obtain that $\rho^\circ(t_*, x_*) \leqslant \rho^*(t_*, x_*) + \alpha + \chi$, and since $\alpha$ and $\chi$ are arbitrarily small, if $q$ is fairly large, we have the inequality

$$\rho^\circ(t_*,\ x_*) \leqslant \rho^*(t_*,\ x_*) \tag{6.6}$$

We obtain the opposite inequality as follows. It follows from (6.1) that when $q \geqslant k(\alpha)$, we have

$$\rho(t_*,\ x_*,\ \Delta_q) \geqslant \rho^*(t_*,\ x_*) - \alpha \tag{6.7}$$

By the definition of $\rho(t_*, x_*, \Delta_q)$ (3.6) it is possible to indicate for any $\zeta > 0$ a non-predicting stochastic program $v_*(\cdot)$ such that whatever the non-predicting stochastic program $u(\cdot)$, the inequality

$$\| w^{(1)}(\cdot,\ u(\cdot),\ v_*(\cdot)) \| \geqslant \rho(t_*,\ x_*,\ \Delta_q) - \zeta \tag{6.8}$$

holds. Then from (6.7) and (6.8) when $q \geqslant k(\alpha)$ we have

$$\| w^{(1)}(\cdot,\ u(\cdot),\ v_*(\cdot)) \| \geqslant \rho^*(t_*,\ x_*) - \alpha - \zeta \tag{6.9}$$

In the stochastic differential equation (3.2) we put $v(\cdot) = v_*(\cdot)$ and construct the sample $u^\circ[t_*[\cdot]\vartheta,\ \omega]$ of the control $u$ along the partitioning steps $\Delta_q\{t_j\}$ on the basis of the optimal strategy $u^\circ(\cdot)$ of the first player, assuming that when $t_j \leqslant t < t_{j+1},\ j = 1, \ldots, q$

$$u^\circ[t_j[\cdot]t_{j+1},\ \omega) = u^\circ(t_j, w(t_j,\ \omega,\ u^\circ(\cdot),\ v_*(\cdot)),\ \varepsilon)$$

This sample, taking (3.4) into account, can be treated as the control formed on the basis of some stochastic non-predicting program $u_*^\circ(\cdot)$. The inequality (6.9) holds for that program. But as regards the property of optimal strategy $u^\circ(t, w, \varepsilon)$ /1/ for any $\chi > 0$ we can indicate $\varepsilon(\chi) > 0$ and $\delta(\varepsilon, \chi) > 0$ such that the measurable application of interference (including any application $v_*[t_*[\cdot]\vartheta,\ \omega)$ formed on the basis of the stochastic non-predicting program $u_*(\cdot)$), the inequality

$$\gamma(w^\circ[t_*[\cdot]\vartheta]) \leqslant \rho^\circ(t_*,\ x_*) + \chi \tag{6.10}$$

will hold, provided $\varepsilon \leqslant \varepsilon(\chi)$ and the partitioning step $\delta(q)$ does not exceed $\delta(\varepsilon, \chi)$. Then, selecting $\varepsilon \leqslant \varepsilon(\chi)$ and assuming in the partitioning that $\Delta_q\{t_j\}\ q > \max\{k(\alpha), [\vartheta - t_0]/\delta(\varepsilon, \chi)\}$, we have besides the inequality (6.9), in conformity with (6.10), for almost applications $v_*[t_*[\cdot]\vartheta,\ \omega)$ the inequality

$$\Big(\int\limits_{[t_*,\vartheta]} |D(t)[w(t, \omega, u_*^\circ(\cdot), v_*(\cdot)) - y(t)]|^2 \mu(dt)\Big)^{1/2} \leqslant \rho^\circ(t_*, x_*) + \chi \tag{6.11}$$

Averaging (6.11) over $\omega \in \Omega$, we obtain

$$\| w^{(1)}(\cdot,\ u_*^\circ(\cdot),\ v_*(\cdot)) \| \leqslant \rho^\circ(t_*,\ x_*) + \chi \tag{6.12}$$

Comparing (6.9) and (6.12) we have $\rho^*(t_*,\ x_*) \leqslant \rho^\circ(t_*,\ x_*) + \chi + \alpha + \zeta$, and, consequently, $\rho^*(t_*, x_*) \leqslant \rho^\circ(t_*, x_*)$. Taking into account (6.6), we find that $\rho^*(t_*,\ x_*) = \rho^\circ(t_*,\ x_*)$. This equation can be derived for any position $\{t_*,\ x_*\}$ and for any sequence of partitioning $\Delta_q\{t_j\}$ for which the limit (6.1) exists. This limit always agrees, as proved, with the payoff of the game $\rho^\circ(t_*,\ x_*)$. This proves that the limit (3.7) exists and is equal to the payoff of the game. The theorem is proved.

From that theorem and inequalitites (6.3) and (6.5) and equations (4.8) and (5.4) we obtain the formula for calculating the payoff of the game.

$$\rho^\circ(t_*, x_*) = \lim_{k\to\infty} \sup_{l(\cdot)\in L[t_*]} [\langle\psi[t_*]\cdot x_*\rangle + \sum_{j=1}^{k} \int\limits_{t_j}^{t_{j+1}} M\{\max_{v\in R} \min_{u\in Q} \langle\psi[\tau, \xi_1, \ldots, \xi_j]\cdot(B(\tau)u + C(\tau)v)\rangle\} d\tau - \int\limits_{[t_*,\vartheta]} \langle G(t)m(t)\cdot y(t)\rangle \mu(dt)] \tag{6.13}$$

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